Calc 12 - Chp 3.1

Note Title 2013-09-23

Derivative of a Function.

We have already looked at derivatives at a point.

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Let's consider the derivative as a function.

If this limit exists for every x in the domain, then we say 'f' is a . There is a second definition (derivative at a point)

This method relies on factor (x-a) from the numerator.

eg) $f(x) = x^2$

$f'(x) =$

subst. & expand

usually binomial exp.

cancel cancel

limit laws

subst. & factor

x-a so we can cancel

cancel

limit laws.

Difference between $f'(x)$ & $f'(a)$:

Use $f'(x)$ for the whole function.

Use $f'(a)$ for a specific value $x = a$.

Either one works if 'a' is remains a variable.

Rationale: We only need to find a derivative function then we can calculate the slope ANYWHERE!

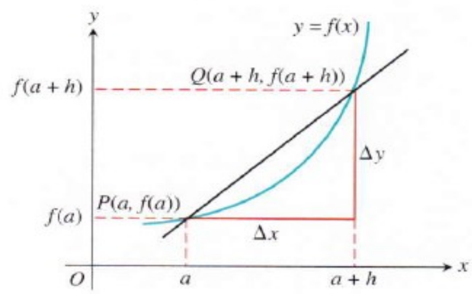


Figure 3.1 The slope of the secant line PQ is

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

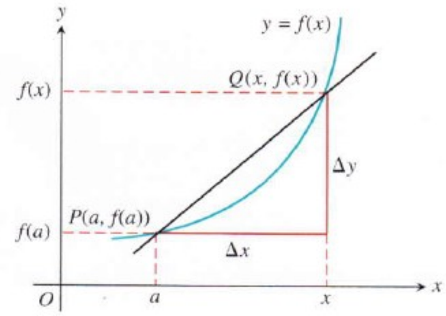


Figure 3.2 The slope of the secant line PQ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}.$$

One sided limits:

Right Side: $\lim\limits_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$

Left Side: $\lim\limits_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$

$\lim\limits_{h \to 0^+} \frac{f(b) - f(b-h)}{h}$

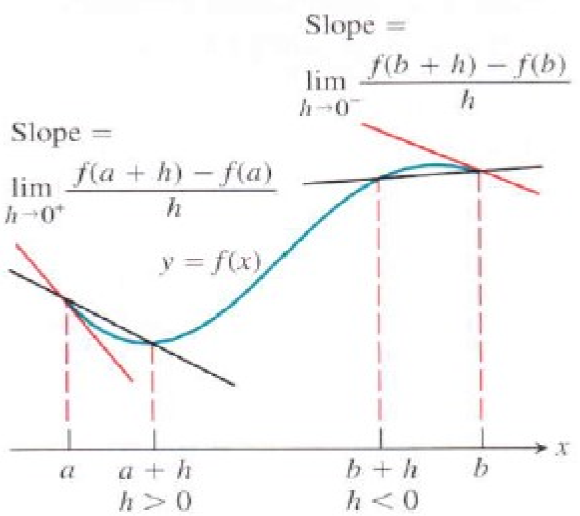
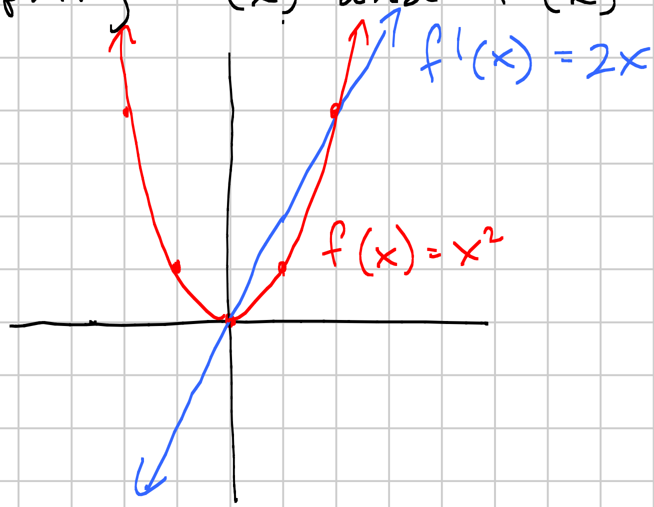


Figure 3.8 Derivatives at endpoints are one-sided limits.

Graphing $f(x)$ and $f'(x)$



$$f'(x) \equiv \frac{d}{dx} f(x) \equiv \frac{dy}{dx}$$

Df

$\dot{y}$

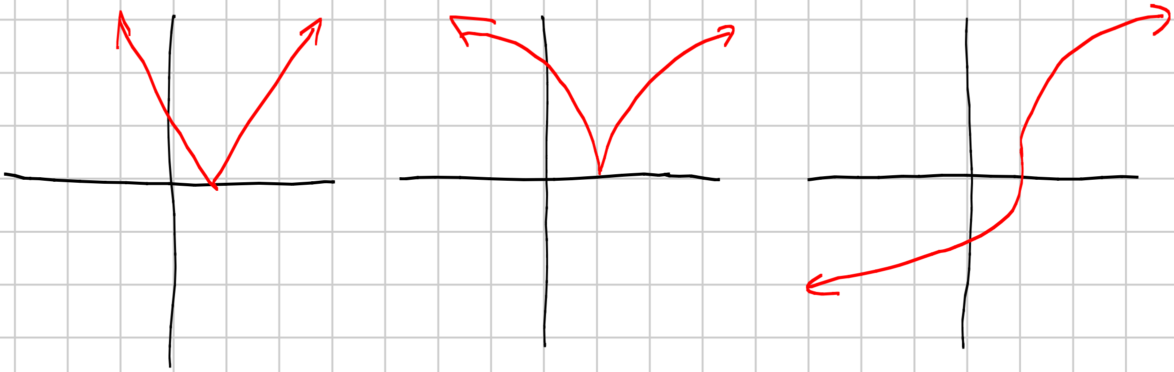
Derivative from a Graph:

1. Look for horizontal tangents of $f(x)$ and plot along the x-axis.
2. Plot holes at sharp corners and cusps of $f(x)$.
3. Determine the slope of lines of $f(x)$ and plot them horizontally.
4. Slopes of parabolas of $f(x)$ are drawn as lines.

HW: pp. 101 – 103: 1, 3, 5, 7, 9, 11, 15, 18
Challenge: 13, 17

Differentiability - in order for a function to be differentiable, it must be relatively smooth.

Here are examples and definitions for non-smoothness:



any discont. will make a fn not differentiable.

Theorem 1: Differentiability Implies Continuity.

If 'f' has a derivative at $x=a$, then 'f' is cont. at $x=a$.

Want to show $\lim_{x \to a} [f(x) - f(a)] = 0$

shows that the limit equals the value

Proof: $\lim_{x \to a} [f(x) - f(a)] =$

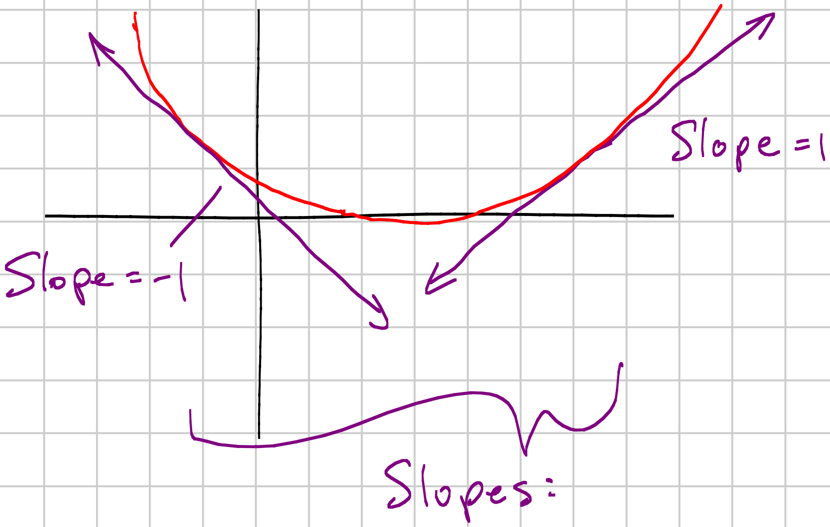
want to make this look like a derivative.

Big note: the converse is NOT always

Theorem 2: IVT for Derivatives

If a & b are any two points in an interval on which f is differentiable, then takes on every value between & .

eg)

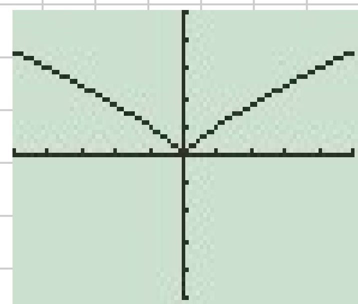


eg) 12)

In Exercises 11–16, the function fails to be differentiable at $x = 0$. Tell whether the problem is a corner, a cusp, a vertical tangent, or a discontinuity.

11. $y = \begin{cases} \tan^{-1} x, & x \neq 0 \\ 1, & x = 0 \end{cases}$

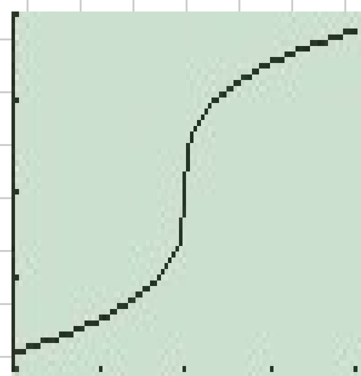
12. $y = x^{4/5}$ **13.** $y = x + \sqrt{x^2} + 2$



18)

In Exercises 17–22, find all values of x for which the function is differentiable.

17. $f(x) = \dfrac{x^3 - 8}{x^2 - 4x - 5}$ **18.** $h(x) = \sqrt[3]{3x - 6} + 5$



HW: p. 111: 1 – 21 (odds), 8
Challenge: 25, 29

Calc 12 - Chp 3.3

Note Title 2013-09-23

Rules for Differentiation.

Rule 1: Deriv of a Constant Fn.

If $f(x) = c$, then $f'(x) = 0$.

Rule 2: Power Rule for Positive Integer Powers of x.

If $f(x) = x^n$, then

eg) $f(x) = x^4 \Rightarrow$

Rule 3: The Constant Multiple Rule

If $f(x) = cx$, then

eg) $f(x) = 7x \Rightarrow$

Rule 4: The Sum and Difference Rule

If $h(x) = f(x) \pm g(x)$, then

eg) $h(x) = x^3 - 4x^2 \Rightarrow$

Rule 5: The Product Rule

If $h(x) = f(x)g(x)$, then $h'(x) = f(x)g'(x) + f'(x)g(x)$

eg) $h(x) = x^3 \cdot 4x^2 \Rightarrow$

Rule 6: The Quotient Rule

If $h(x) = \dfrac{f(x)}{g(x)}$, then $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

eg) $h(x) = \dfrac{4x^3}{7x^2} \Rightarrow$

Rule 7: Power Rule for Negative Integer Powers of x

If $f(x) = x^n$ and $n \neq 0$, then $f'(x) = n \cdot x^{n-1}$

eg) $f(x) = 1/x^3$

We can also take derivatives of derivatives called higher order derivatives. We have been doing first derivatives. If we take the derivative again, it is called the second derivative ($f''(x)$). Successive derivatives are called third ($f'''(x)$), fourth ($f^{(4)}(x)$), etc.

eg) $f(x) = x^7$

How many more non-trivial derivatives can we take?

eg) #24)

24. Suppose u and v are functions of x that are differentiable at $x = 2$ and that $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, and $v'(2) = 2$. Find the values of the following derivatives at $x = 2$.

(a) $\dfrac{d}{dx}(uv)$ **(b)** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$

(c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **(d)** $\dfrac{d}{dx}(3u - 2v + 2uv)$

a)

b)

c)

d)

HW: pp. 120 - 121: 1 - 29 (odds), 28

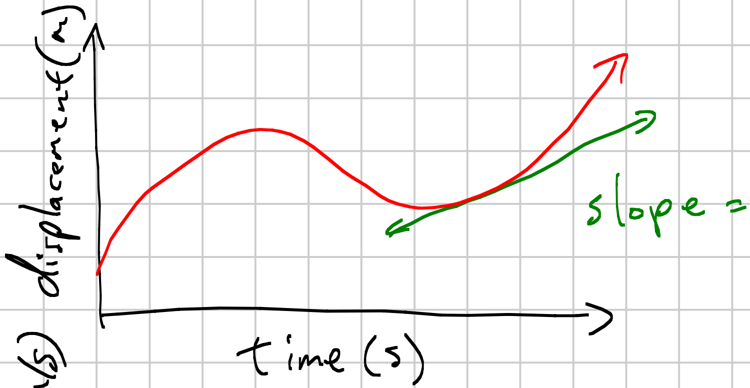
Challenge: 31, 33, 35

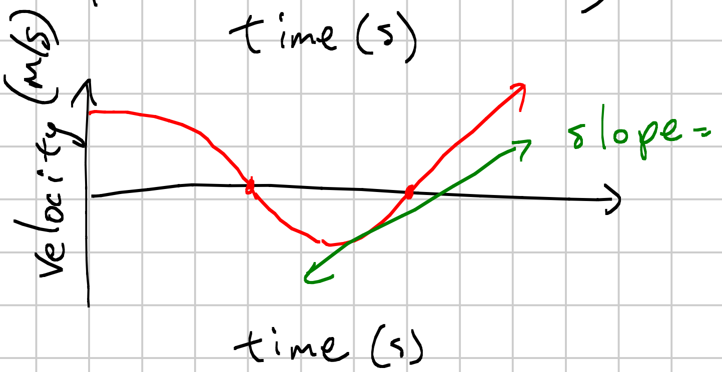
Calc 12 - Chp 3.4

Velocity and Other Rates of Change

$$m = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This derivative is the instantaneous rate of change.





displacement =
distance =
velocity =
speed =
acceleration =
jerk = $j(t)$ =

eg) A drag car's displacement (in feet) is given by the formula: $f(t) = 51t^2 - 2.5t^3$

Find the velocity and acceleration as a function of time.

Find the velocity after 6 seconds. (in mph)

When does the dragster reach maximum vel?

Find the maximum velocity. (in mph)

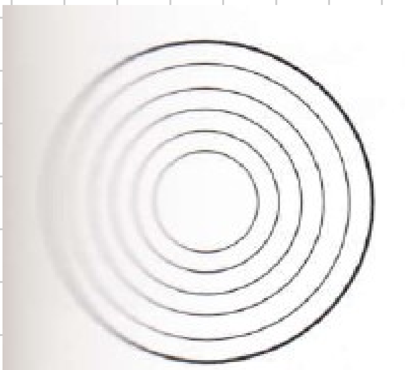
5280 ft = 1 mi.

$v(t) =$

$a(t) =$

$v(6) =$
$a(t) =$

Tree Growth



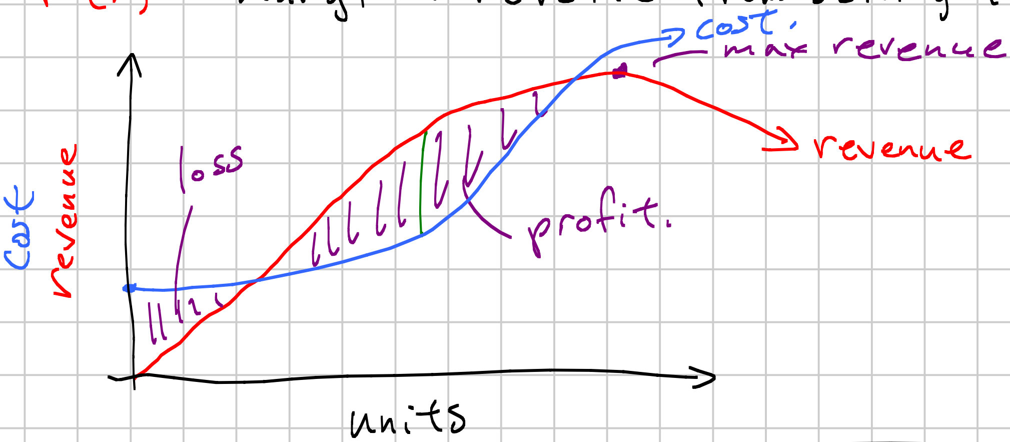
$\Delta r \approx c$

$\Delta r \downarrow$

← Better Model. Why?

Economics.
We can look at these formulas in mass production.
It doesn't make sense on the small scale.

$c(x)$ = cost to produce x units
$r(x)$ = revenue from selling x units
$c'(x)$ = marginal cost to make the x^{th} unit.
$r'(x)$ = marginal revenue from selling the x^{th} unit.



Anytime a question asks for a change of quantity, take the derivative!

eg)

10. *Marginal Cost* Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

(a) Find the average cost of producing 100 washing machines.

(b) Find the marginal cost when 100 machines are produced.

(c) Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

Average cost must use $\dfrac{C(x)}{x}$

not $\dfrac{C(x) - C(0)}{x}$

a)

b)

c)

Particle Motion *Work in groups of two or three.* The position (x-coordinate) of a particle moving on the line $y = 2$ is given by $x(t) = 2t^3 - 13t^2 + 22t - 5$ where t is time in seconds.

(a) Describe the motion of the particle for $t \geq 0$.

(b) When does the particle speed up? slow down?

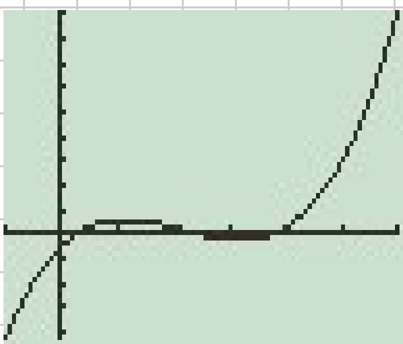
(c) When does the particle change direction?

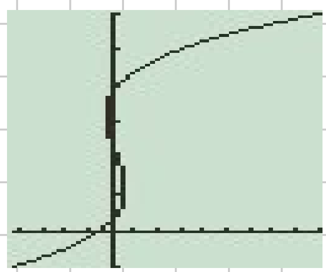
(d) When is the particle at rest?

(e) Describe the velocity and speed of the particle.

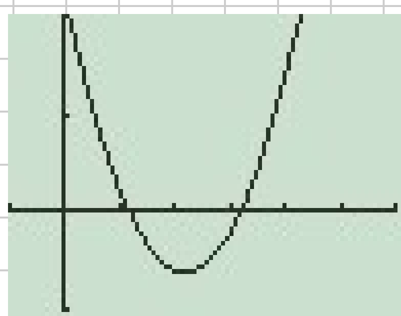
(f) When is the particle at the point (5, 2)?

a)





b)



c)

d)

e)

f)

HW: pp. 129 - 133 : 1, 3, 5, 9, 13, 23, 29, 6
Challenge : 15, 16

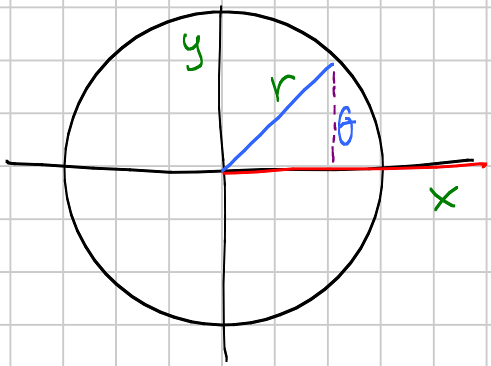
Calc 12 - Chp 3.5

Note Title 2013-09-23

Trigonometric Functions in a Big nutshell.

Trigonometry is for triangles.
Trigonometric functions model much more such as waves - water, light, sound; alternating current; other physical rhythms.

Use Unit Circle

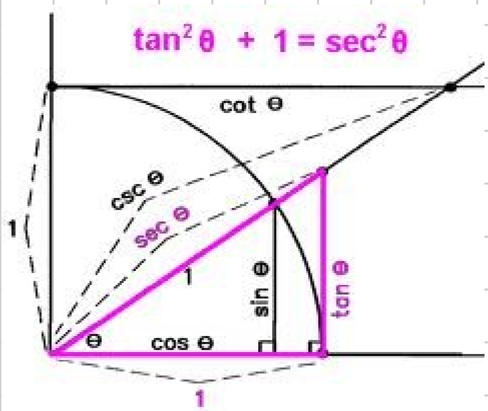


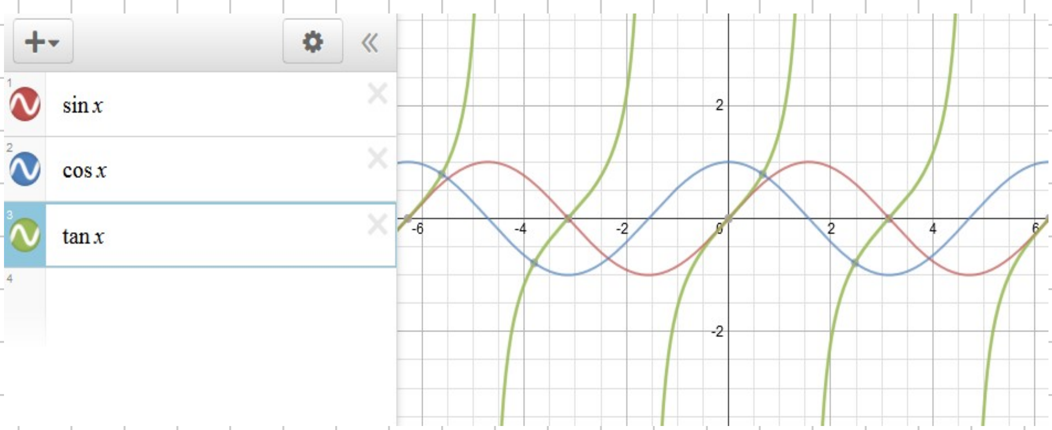
θ - is in standard position when measured from initial side to terminal side.

Radians – ratio of arc length to radius.
- must use radians in Calc.

r=1

$\sin\theta = \frac{opp}{hyp} =$	$\csc\theta =$	Cosecant
$\cos\theta = \frac{adj}{hyp} =$	$\sec\theta =$	Secant
$\tan\theta = \frac{opp}{adj} =$	$\cot\theta =$	





Derivatives of Trig Fns. - Trig identities are a MUST to find derivatives through limits!

$\sin(-\theta) = -\sin\theta$
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$\cos(-\theta) = \cos\theta$
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$\sin^2\theta + \cos^2\theta = 1 \qquad \sec^2\theta = 1 + \tan^2\theta$
rest on p. 580.

Recall:

$$\left.\lim_{h\to 0} \frac{(\cos h - 1)}{h} = 0 \quad \& \quad \lim_{h\to 0} \frac{\sin h}{h} = 1 \right\} \text{can't prove yet}$$

Find derivative of $f(x) = \cos x$

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$\frac{d}{dx}\sin x = \cos x$$

shortcut for: $f(x) = \sin x$, $f'(x) = \cos x$

Notice:

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

eg) Find the velocity and acceleration of $f(x) = \sin x \cdot \sin x$

eg) 12. Find an equation of the line normal to the graph of $y = (\tan x)/x$ at $x = \pi/4$.

HW: p. 140: 1 – 23 (odds), 18
Challenge: 25, 29, 33

Calc 12 - Chp 3.6

Note Title 2013-09-23

Chain Rule - used for composite fns.

Intro to Composite Fns - take the output of one fn and use it as an input to another fn. This can happen more than once.

This is common in business.

$x = \text{Cost}(p)$
$y = \text{DistributionPrice}(x)$
$z = \text{WholesalePrice}(y)$
$u = \text{RetailPrice}(z)$
$v = \text{Tax}(u)$
$v = T(R(W(D(C(p)))))$
$= (T \circ R \circ W \circ D \circ C)(p)$

$f(g(x)) = (f \circ g)(x)$
"f of g of x"

Notice the first operation is the innermost
Just like BEDMAS

Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$ or $\frac{df}{dg} \cdot \frac{dg}{dx}$

Simple
eg) $f(x) = (x^3)^5$ $\Rightarrow$

simplify then differentiate.

Medium:
eg) $f(x) = \sin(x^3)$

chain rule solves this; can't simplify first!

Power Chain Rule: $f(x) = [g(x)]^n$

eg) $f(x) = \sin^2 x$

Different Methods, Same answer

or

$f(x) = \sin x \cdot \sin x$

Hard

eg) $f(x) = \sin(\sin x^2)$

Another way to view chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
(Substitution)

$y = \cos(x^4)$

$\boxed{\text{the du's do not cancel}}$

The case for radians:

$f(x) = \sin x$ (in degrees)

$= \sin\left(\frac{\pi x}{180}\right)$

$f'(x) = \cos\left(\frac{\pi x}{180}\right) \frac{d}{dx}\left(\frac{\pi x}{180}\right) = \cos\left(\frac{\pi x}{180}\right)\left(\frac{\pi}{180}\right)$

$f''(x) = -\frac{\pi}{180} \sin\left(\frac{\pi x}{180}\right) \frac{d}{dx}\left(\frac{\pi x}{180}\right) = -\left(\frac{\pi}{180}\right)^2 \sin\left(\frac{\pi x}{180}\right)$

So it gets messy, whereas in radians.

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$\vdots$

#34) Find $(f \circ g)'(x)$.

$f(u) = 1 - \frac{1}{u}$

$g(x) = \frac{1}{1-x}$

#36 $f(u) = u + \frac{1}{\cos^2 u} = u + (\cos u)^{-2}$, $x = 1/4$

$g(x) = \pi x$

Hw: pp. 146-147: 1-33 (odds), 32

Challenge: 35, 37

Calc 12 - Chp 3.7

Note Title 2013-09-23

Implicit Differentiation - some equations are hard to express in terms of y. So an easier way to find the derivative is to use implicit differentiation.

eg) $x^3 + y^2 = 15$. Find dy/dx @ $(2, \sqrt{7})$

differentiate each term wrt x
with respect to

y is a function of x
so use chain rule
isolate $\frac{dy}{dx}$

Very General Steps for Implicit Differentiation:

1. differentiate each term wrt x
2. collect dy/dx terms on LHS.
3. factor out dy/dx
4. isolate dy/dx.

Note: can use y' instead of dy/dx

---

eg) find y' of $x^2y^2 + x^3 + y^4 = 20$

eg) Find y'' of first example: $y' = \dfrac{-3x^2}{2y}$

quotient rule

Could try implicit

$2yy' = -3x^2$

$2y'y' + 2yy'' = -6x$

$2yy'' = -6x - 2(y')^2$

$2yy'' = -6x - 2\left(\dfrac{-3x^2}{2y}\right)^2$

$2yy'' = -6x - \dfrac{18x^4}{4y^2}$

$y'' = \dfrac{-24xy^2 - 18x^4}{8y^3}$

$= \dfrac{-12xy^2 - 9x^4}{4y^3}$

---

eg) With trig: Find y' of $7x^2 + 3x\cos y = xy$

HW: p. 155: 3, 9, 11, 17, 19, 23 – 29 (odds), 28

Challenge: 22

Calc 12 - Chp 3.8

Note Title 2013-09-23

Derivatives of Inverse Functions.

p. 158 Theorem 3 - Derivative of Inverse Fns.

If is differentiable at every point of an interval and is never on , then has an inverse and is differentiable at every point of the interval ,

Big picture: If we can differentiate, it stands that we can take the reciprocal of the derivative.

eg) $y = \sin^{-1} x$

$$\sin y = x$$

$$\frac{d}{dx} \sin y = \frac{d}{dx} x$$

$$y' \cos y = 1$$

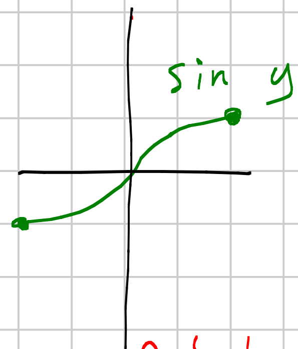
$$y' = \frac{1}{\cos y}$$

inverse

implicit diff

differentiate

isolate y', but must solve in terms of 'x'



p. 161

Inverse Function – Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$

$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$

$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

Calculator Conversion Identities

$$\sec^{-1} x = \cos^{-1}(1/x)$$

$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$

$$\csc^{-1} x = \sin^{-1}(1/x)$$

Derivatives of Inverse Trig Fns.

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$$

eg) $f(x) = \sin^{-1}(x^3)$ Find $f'(.2)$

Let $u = x^3$:

$\frac{du}{dx} = 3x^2$

eg) $f(x) = \cot^{-1}\frac{1}{x} - \tan^{-1}x$ Find $f'(x)$

$f'(x) =$

eg) $f(t) = \dfrac{1}{\cos^{-1}(3t^2)}$ Find $f'(.4)$

quotient rule & chain rule

or use power chain rule.

Power Rule: $f(t) = [\cos^{-1}(3t^2)]^{-1}$

$f'(t) = -[\cos^{-1}(3t^2)]^{-2} \dfrac{d}{dx}(\cos^{-1}(3t^2))$

$= -[\cos^{-1}(3t^2)]^{-2} \dfrac{-1}{\sqrt{1-(3t^2)^2}} \dfrac{d}{dt}(3t^2)$

= same as above.

Derivative of Inverse Function @ $f(a)$: $\left(\dfrac{d f^{-1}}{dx}(\overset{y}{f(a)})\right) = \dfrac{1}{f'(\underset{x}{a})} \neq \left(\dfrac{d}{dx} f^{-1}(a)\right)$

HW: p. 162: 1–19 (odds), 10

Challenge: $y = \cos^{-1} x$. Derive y' using implicit differentiation

21

$\csc^{-1} x = \sin^{-1} 1/x$ identity

Derivatives of Exponential and Logarithmic Fns

Exponential Functions - looks like powers.
However, there is a major difference; the variable is in exponent rather than the base and the base must be

The domain for exp fns is all reals.
Range for untransformed exp fn is $y>0$
Untransformed exp fns are ALWAYS INCREASING!

Definition of a Logarithm - The logarithm of a number is an exponent. $\log_b c$ is the power to which b is raised to get c. The base of the logarithm is the same as the base of the power.
When , then , where .

Exponentials and Logarithms are inverses of each other!
eg) $4^3 = 64$ $\qquad \log_4 64 = 3$

$2^4 \cdot 2^5 = 2^9$ $\qquad \log 2^4 + \log 2^5 =$
$4 \cdot 8 = 32$ $\qquad \log 4 + \log 8 =$
$\log_a x + \log_a y = \log_a xy$

$\frac{3^7}{3^4} = 3^3$ $\qquad \log 3^7 - \log 3^4 =$
$28/7 = 4$ $\qquad \log 28 - \log 7 =$
$\log_a x - \log_a y = \log_a (x/y)$

$(5^2)^4 = 5^8$ $\qquad \log (5^2)^4 =$
$\log 6^3 =$
$\log_a x^k = k \log_a x$

$\log_a 1 = 0$

$\log_a a = 1$

$\log_a a^b = b$

change of base

$\log_b a = \frac{\log_c a}{\log_c b}$
$= \frac{\log a}{\log b}$

e^x is a most interesting fn.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

$$\frac{d}{dx} e^x = e^x$$

another way to look at it

Property: $\displaystyle\lim_{h\to 0} \frac{e^h - 1}{h} = 1$ or $\displaystyle\lim_{x\to 0} \frac{e^x - e^0}{x - 0} = 1$

$f(x) = e^x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

difficult? No.

Chain Rule:

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

$\ln x = \log_e x$

Derivative of a^x, $a > 0$ & $a \neq 1$

$$\frac{d}{dx} a^x =$$

exp rules.

use chain rule.

Chain Rule: $\displaystyle\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$ $\quad a > 0$ & $a \neq 1$

Derivative of $\ln x$, $x > 0$

$y = \ln x$

(1)

exponentiate both sides.
diff wrt x

isolate y

substitute from (1)

Chain Rule: $\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$

Derivative of $\log_a x$, $a > 0$, $a \neq 1$, $x > 0$

$\frac{d}{dx} \log_a x =$

exp rules.

Chain Rule: $\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx}$ $a > 0$ & $a \neq 1$

eg) $y = e^{-x/4}$ Find y'

eg) $y = 9^{-x}$ Find y'

eg) $y = (\ln x)^2$

eg) $y = \log_{10} \sqrt{x+1}$

eg) $y = x^{\tan x}, \quad x > 0$

HW: p. 170: 1 - 37 (every other odd), 28
Challenge: 43
Hint: Take ln & implicit differentiation.