Calc 12 - Chp 4.1

Note Title 2013-09-23

In many real-life applications, we are interested in the maximum and/or minimum values called extrema. These values are also important mathematically. In engineering, they are interested in the maximum load that a structure is capable of supporting or how much pressure a submarine can take before getting crushed by water. When graphing, extrema usually signal a change in the rate of change, so this is significant.

Definition: Absolute Extreme Values.

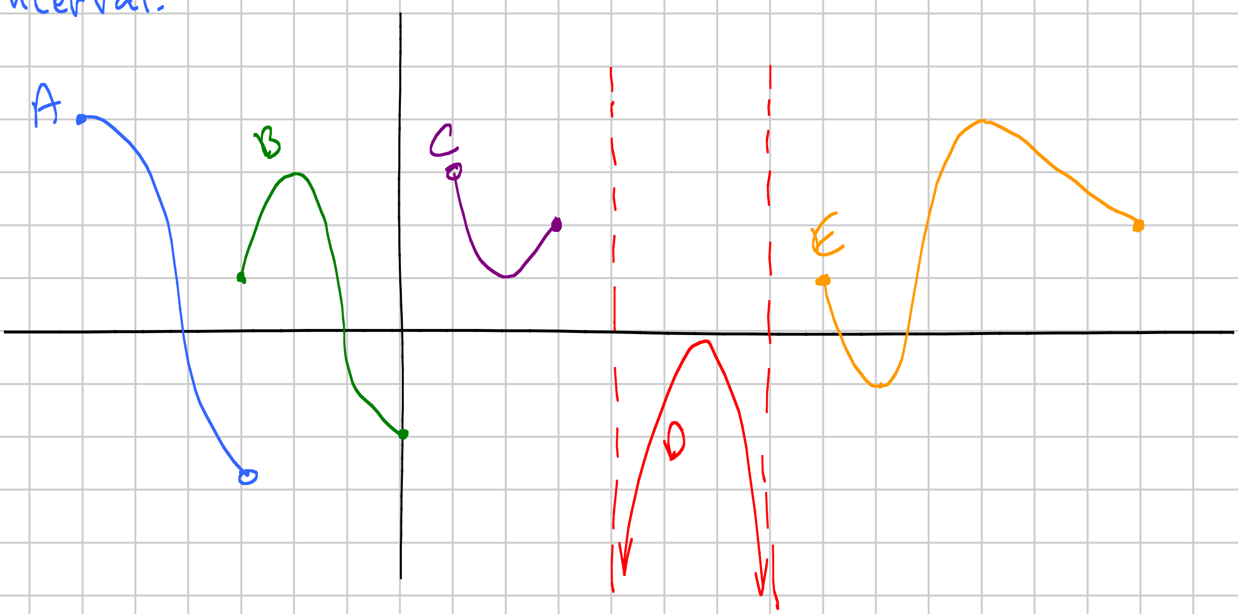
Let f be a function with domain D. Then $f(c)$ is the

(a) on D iff $f(c) \geq f(x)$ $\forall x \in D$.

(b) on D iff $f(c) \leq f(x)$ $\forall x \in D$.

Theorem 1: Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval.



Definition: Local Extreme Values.

Let c be an interior point of the domain of f. Then $f(c)$ is a

(a) at c iff $f(c) \geq f(x)$ for x in some open interval containing c.

(b) at c iff $f(c) \leq f(x)$ for x in some open interval containing c.

Absolute extremum are also local extremum.

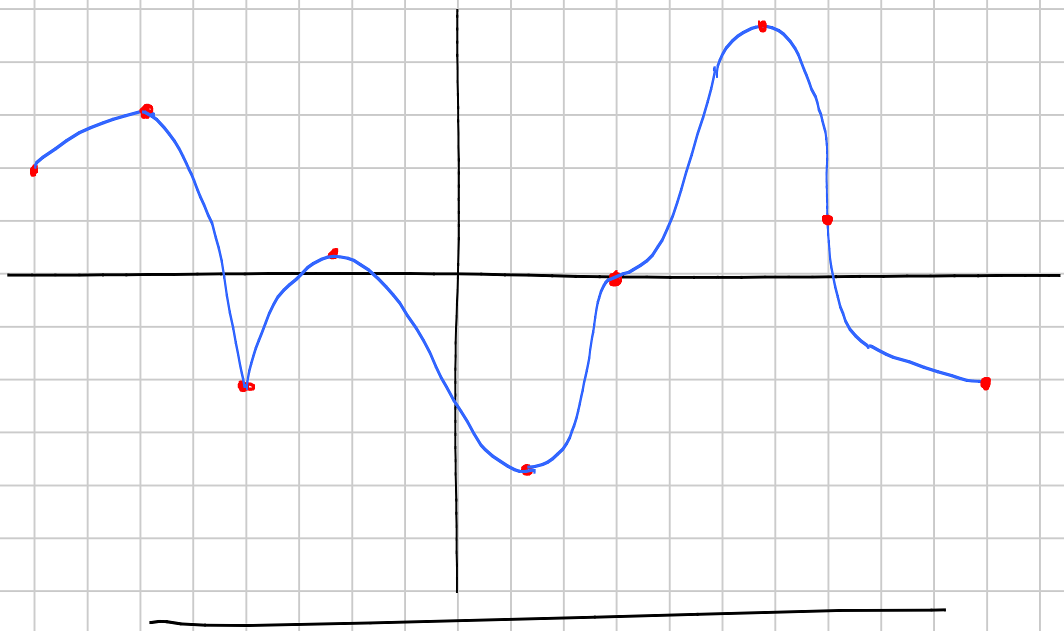
Theorem 2: Local Extrema Values.

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then $f'(c) = 0$.

Definition: Critical Point.

A point in the interior of the domain of a function f at which $f' = 0$ or f' D.N.E. is a C.P. of f.

Extreme values occur only at C.P.'s and endpoints.



Finding Extrema Part 1

1. Compute f' and find C.P.'s ($f'=0$ or f' D.N.E.)
2. Check graphically if C.P. is an extremum.
3. Check endpoints (if they exist) are extrema.

Finding Extrema Analytically

use (1) above. f' DNE can occur on boundaries of piecewise functions.

eg) #12) $g(x)=e^{-x}, \quad -1 \le x \le 1$

eg) #14) $k(x)=e^{-x^2}, \quad -\infty < x < \infty$

eg) #18) $f(x)=x^{3/5}, \quad -2 < x \le 3$

H.W. pp. 184-185: 1-29 odds, 46, 48

Challenge 41, 43

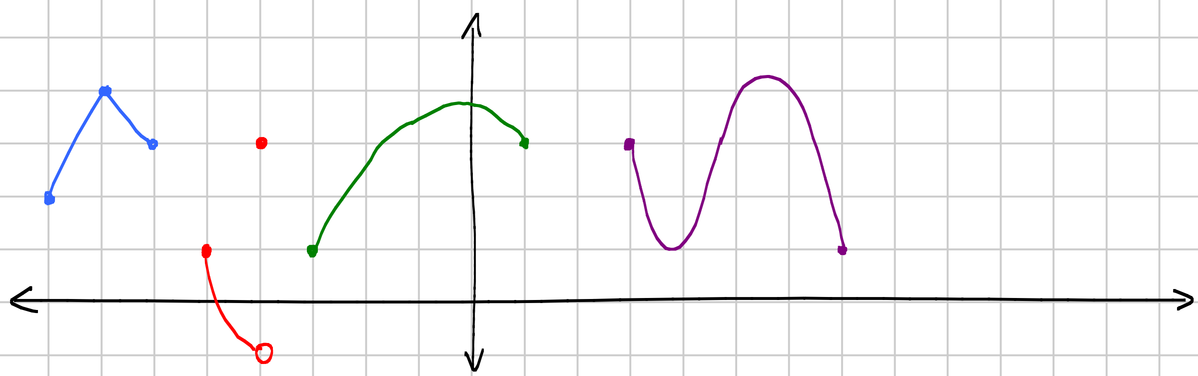
Calc 12 - Chp 4.2

Note Title 2013-09-23

Recall IVT (Intermediate Value Theorem) that if f is continuous, then $\exists c \in (a,b)$ s.t. $f(a) < f(c) < f(b)$. It is an existence theorem, so it doesn't say what the exact value is.

Theorem 3: Mean Value Theorem.
If $y = f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then

This is also an existence theorem. Recall that 'mean' is the arithmetic average, but we are looking at the average rate of change, so the formula looks a little different than arithmetic mean. More restrictive than IVT because it needs to be differentiable. But we can usually solve 'c'.



Definitions: Increasing Fn and Decreasing Fn
Let f be a fn on an interval I and x_1 & $x_2 \in I$, $x_1 \neq x_2$.
1. f is an Increasing Fn on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
2. f is a Decreasing Fn on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

We don't prove definitions, we understand and use them.

Corollary 1: Increasing and Decreasing Fns.

Let f be continuous on $[a,b]$ and differentiable on (a,b).

1. If $f' > 0$ at each point (a,b), then f incr on $[a,b]$.
2. If $f' < 0$ at each point (a,b), then f decr on $[a,b]$.

Proof: Use definition: Let x_1 & $x_2 \in [a,b]$, $x_1 \neq x_2$.

Use MVT: $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ for some $c \in (x_1, x_2)$

Think about IVT. What has to happen for an incr fn to become a decr fn and vice versa?

Corollary 2: Fns with $f' = 0$ are constant.

If $f'(x) = 0$ on an interval I, then

Corollary 3: Fns with the same derivative differ by a constant.

If $f'(x) = g'(x)$ at each point of an interval I, then

s.t.

eg) $f(x) = \sin x$,

$f'(x) = \cos x$,

Definition: Antiderivative.

A fn $F(x)$ is an antiderivative of a fn $f(x)$

in the domain of f. The process of finding an antiderivative is called

eg) 28) $f'(x) = \sin x$ $\Rightarrow$

eg) $f'(x) = 8x^3$ $\Rightarrow$

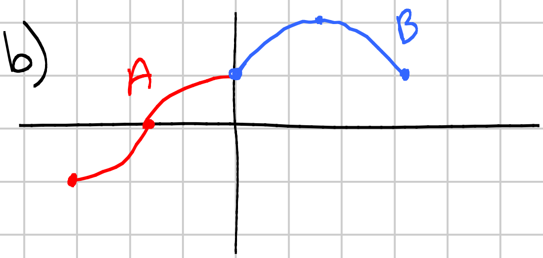
p. 192

18) $f(x) = \ln(x-1)$, $x \in [2,4]$ MVT

24) $f(x) = \begin{cases} \cos x, & -\pi \le x < 0, \\ 1+\sin x, & 0 \le x \le \pi, \end{cases}$ on $[-\pi, \pi]$

a) not differentiable at $x=0$

b)



c)

HW: p. 192: 1-15 odds, 25-29 odds, 38, 39
Challenge: 31

Calc 12 - Chp 4.3

Note Title 2013-09-23

Using f' & f" to Graph f

Theorem 4:

At a critical point 'c':

1. If $f' > 0$ for $x < c$ and $f' < 0$ for $x > c$, then f has a local max at c. $f'(c) = 0$ or D.N.E.
2. If $f' < 0$ for $x < c$ and $f' > 0$ for $x > c$, then f has a local min at c. $f'(c) = 0$ or D.N.E.
3. If f' does not change sign at c, then f has no local extreme at c. $f'(c) = 0$ or D.N.E.

At left endpoint 'a':

If $f' < 0$ for $x > a$, then f has a local max at a.

At right endpoint 'b':

If $f' > 0$ for $x < b$, then f has a local max at b.

Definition: Concavity

The graph of a differentiable fn $y = f(x)$ is

(a) concave up on an open interval I if y' is incr on I.

(b) concave down on an open interval I if y' is decr on I.

Concavity Test

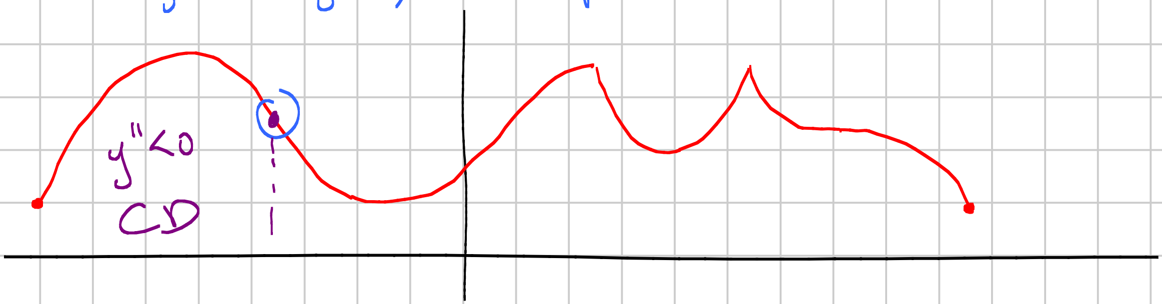
The graph of a twice diff fn $y = f(x)$ is

(a) concave up on any interval where $y'' > 0$

(b) concave down on any interval where $y'' < 0$

Point of Inflection

A point on a graph where y'' changes signs (concavity changes) is a point of inflection



Theorem 5: Second Derivative Test for Local Extrema

1. If $f'(c)=0$ and $f''(c)<0$, then f has a local max @ $x=c$.
2. If $f'(c)=0$ and $f''(c)>0$, then f has a local min @ $x=c$.

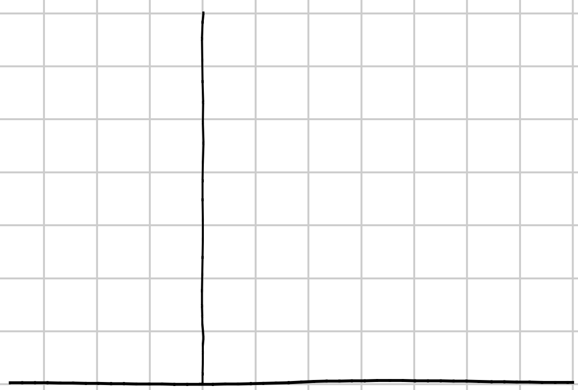
eg)

Exploration 2 Finding f from f' and f''

A function f is continuous on its domain $[-2, 4]$, $f(-2) = 5$, $f(4) = 1$, and f' and f'' have the following properties.

x	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$2 < x < 4$
f'	$+$	does not exist	$-$	0	$-$
f''	$+$	does not exist	$+$	0	$-$

1. Find where all absolute extrema of f occur.
2. Find where the points of inflection of f occur.
3. Sketch a possible graph of f.



eg) $y = \frac{1}{4}x^4 + x^3 - \frac{x^2}{2} - 3x + 2$

$y' = x^3 + 3x^2 - x - 3$

$y'' = 3x^2 + 6x - 1$

$y' = 0$ @ $x = \pm 1, -3$

$y'' = 0$

$x = \dfrac{-6 \pm \sqrt{6^2 - 4(3)(-1)}}{2(3)}$

$= \dfrac{-1 \pm \sqrt{48}}{6}$

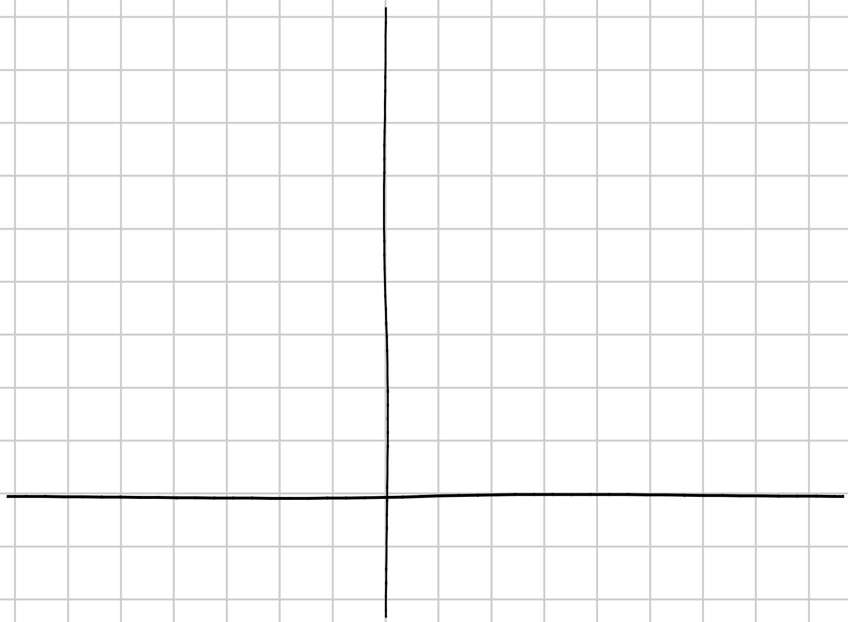
Factor property: $\pm 1, \pm 3$

$+1$ works

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -1 & -3 \\ & \downarrow & -1 & -2 & 3 \\ \hline & 1 & 2 & -3 & 0 \end{array}$$

$x^2 + 2x - 3 = (x+3)(x-1)$

$(x+1)$



Warning: Second derivative test may fail: $f''(x)=0$ or DNE for determining extrema, so use first derivative test instead.

eg) $f(x)=x^4$; $f'(x)=4x^3$; $f''(x)=12x^2$;
$f'(0)=0$; $f''(0)=0$; $f'(-1)=-4$; $f'(1)=4$;
so $f(0)$ is a minimum.

Hw. pp. 203-205: 1-21 odds, 37-41 odds, 4
Challenge: 31

Calc 12 – Summary for Curve Sketching

Domain: determine the domain for $f(x)$.

Intercepts: evaluate $f(0)$ for the y-intercept and set $f(x)=0$ for the x-intercept (also called finding the roots).

Asymptotes: check for $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$ for horizontal asymptotes. Check for NPV in denominators for vertical asymptotes, that is,

$$\lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$
$$\lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

Check for slant asymptotes using $\lim_{x \to \infty}\left[f(x) - (mx + b)\right] = 0$. For rational functions, this occurs when the numerator is one degree more than the denominator, such as

$$f(x) = \frac{x^3}{x^2 + 1}.$$

Intervals of Increase or Decrease: Compute $f'(x)$ and find the intervals where $f'(x)>0$ and $f'(x)<0$. Can also use $f''(x)=0$ for <u>inflection points</u>.

Local Maximum and Minimum Values: Find the critical numbers where $f'(c)=0$ or $f'(c)$ DNE. Use the First Derivative Test to check for a change of sign to determine local min/max; if f' changes from positive to negative at c, then $f(c)$ is a local max; if f' changes from negative to positive at c, then $f(c)$ is a local min. Or you can use the Second Derivative Test provided that $f''(c)\neq 0$ *and* $f''(c)$ *exists*; if $f''(c)>0$ implies a min or $f''(c)<0$ implies a max.

Concavity and Points of Inflection: Compute $f''(x)$ in each inflection interval to determine concavity.

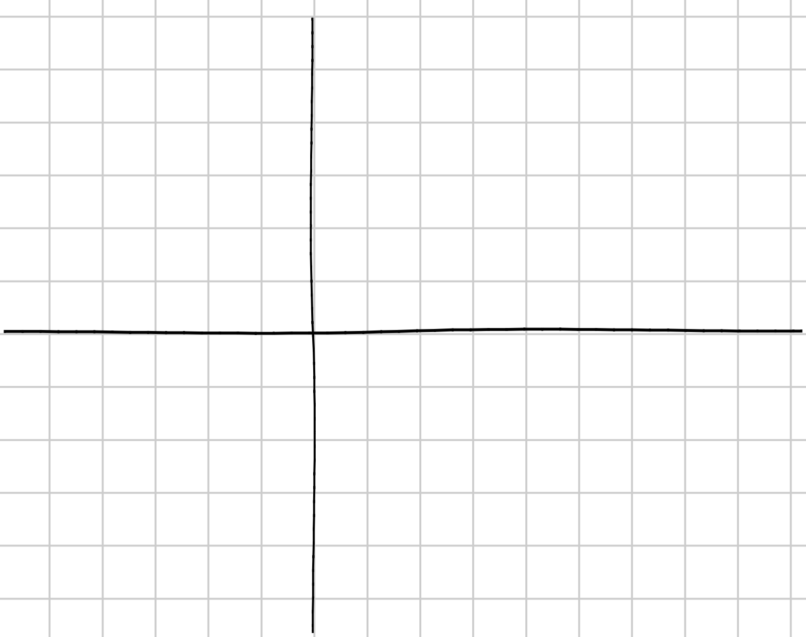
Sketch the Curve: Use the above info to make the sketch. Not all functions will generate all of the above info.

Use 2nd deriv test only if 2nd deriv is easy to calculate. So if using quotient rule for 1st deriv, then probably not a good idea to do the 2nd deriv test.

40. $s(t) = 3t^2 - 2t^3$

domain : $t \in \mathbb{R}$

range : $s \in \mathbb{R}$



HW: pp. 204-205 :
fully graph
$s(t) = t^3 - 3t + 3$
$s(t) = t^2 - 4t + 3$

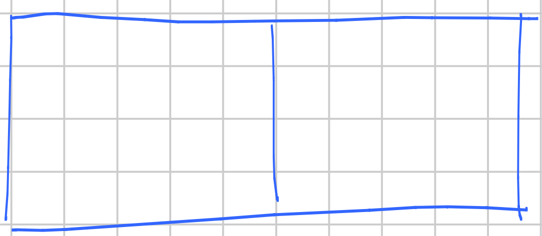
Calc 12 - Chp 4.4

Modelling and Optimization - Word Problems.

1. Determine if min or max.
2. If you are given a formula, use it.
3. Make sure the formula has the same units as the question.
4. If making your own function, check units to match the problem and make a drawing if possible.

5. Determine if answers are in the domain or are valid.

eg) p. 214 #10
216 m^2 rectangular pea patch divided into 2.
Dimension of outer fence and amount of fencing



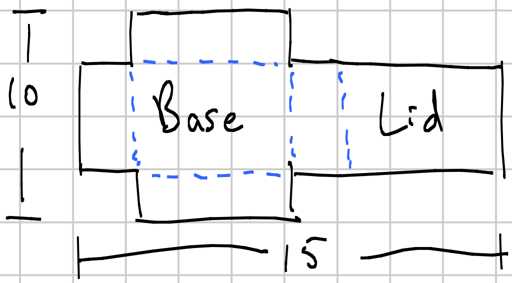
Area =

Perimeter =

Minimize $P(x)$. Find C.P.'s of $P(x)$

eg) p. 214 #18

Cardboard is 10 in × 15 in



a)

b)

c)

d)

$C(x) \equiv$ cost

$R(x) \equiv$ revenue

$P(x) \equiv$ profit

$= R(x) - C(x)$

$C'(x) \equiv$ marginal cost

$R'(x) \equiv$ marginal revenue

$P'(x) \equiv$ marginal profit

Maximum Profit Theorem: It occurs when

since $P'(x) = R'(x) - C'(x)$ and we want $P'(x) = 0$.

Minimize Average Cost Theorem: Use

to find min avg cost (if it exists).

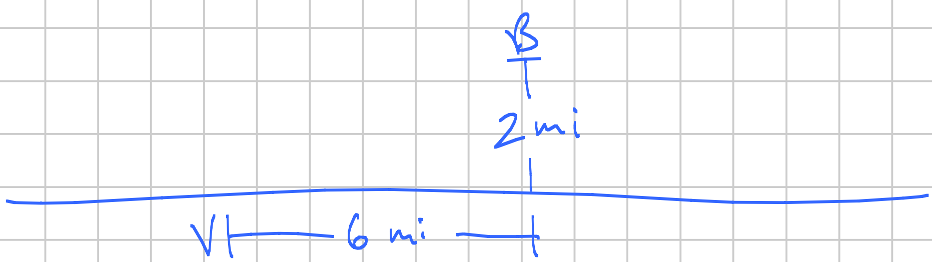
eg) p. 219 #52

$C(x) = x^3 - 20x^2 + 20000x$

Minimize cost per unit: $u(x) = c(x)/x$

p. 215 #20

20. *Quickest Route* Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?



time $= t(x) =$

$t''(x)$

1st deriv test:

HW: pp. 214-215: 5, 7, 9, 11, 15, 17, 22
Challenge: 29, 31

Linearization and Newton's Method

Recall tangent line at a point (a) on a curve: f(x)

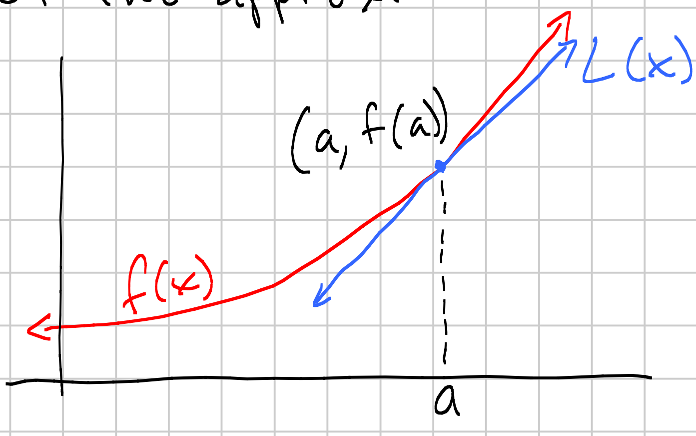
$y = f(a) + f'(a)(x-a)$

Nothing new except making it a function:

Definition: Linearization
If 'f' is differentiable at $x=a$, then the approx. function

is a linearization of 'f' at 'a'.

The approx $f(x) \approx L(x)$ is the standard linear approx of 'f' at 'a'. The point $x=a$ is the center of the approx.

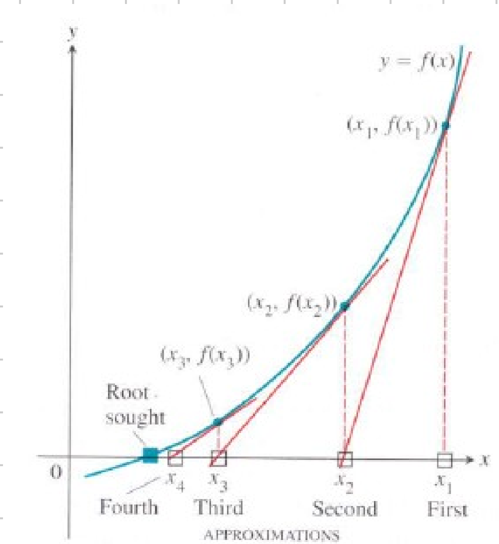


L(x) is a good approx when 'x' is close to 'a'

eg) Find the linearization of $y = \tan x$ at $x = \pi/4$

eg) Find an approx of $\sqrt{9.1}$ without a calculator.

Newton's Method – A direct application of linearization. It is used to find zeroes of any function.



In general, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

We stop when we are as close as we need.

Q: When won't Newton's Method work?

eg) Find $\sqrt[6]{2}$ using Newton's Method to 3 decimals.

$X = \sqrt[6]{2}$

$X^6 = 2$ Because 'x' isn't meaningful.

$X^6 - 2 = 0$ because NM finds zeroes.

```
PROGRAM:NM
:A-Y1(A)/Y2(A)
:Ans→A
:Disp A
:Stop
:
```

```
Plot1 Plot2 Plot3
\Y1=X^6-2
\Y2=6X^5
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
1/Ans
                 32
1→A
                  1
prgmNM
        1.166666667
               Done
```

Guess: $x_1 = 1$

```
               Done
prgmNM
        1.126443678
               Done
prgmNM
        1.122497067
               Done
```

$$X_2 = 1 - \frac{1^6 - 2}{6(1)^5} \sim 1.1\dot{6}$$

$$X_3 = 1.1\dot{6} - \frac{(1.1\dot{6})^6 - 2}{6(1.1\dot{6})^5} \sim 1.12644$$

$$X_4 = 1.12644 - \frac{(1.12644)^6 - 2}{6(1.12644)^5} \sim 1.12249$$

$$X_5 = 1.12249 - \frac{(1.12249)^6 - 2}{6(1.12249)^5} \sim 1.12246$$

We know this is accurate to at least 4 decimal places. This method is used to find zeroes and intersections on your calculator.

```
               Done
prgmNM
        1.122462051
               Done
prgmNM
        1.122462048
               Done
```

HW: p. 229: 1-15 odds, 2

Challenge: none.

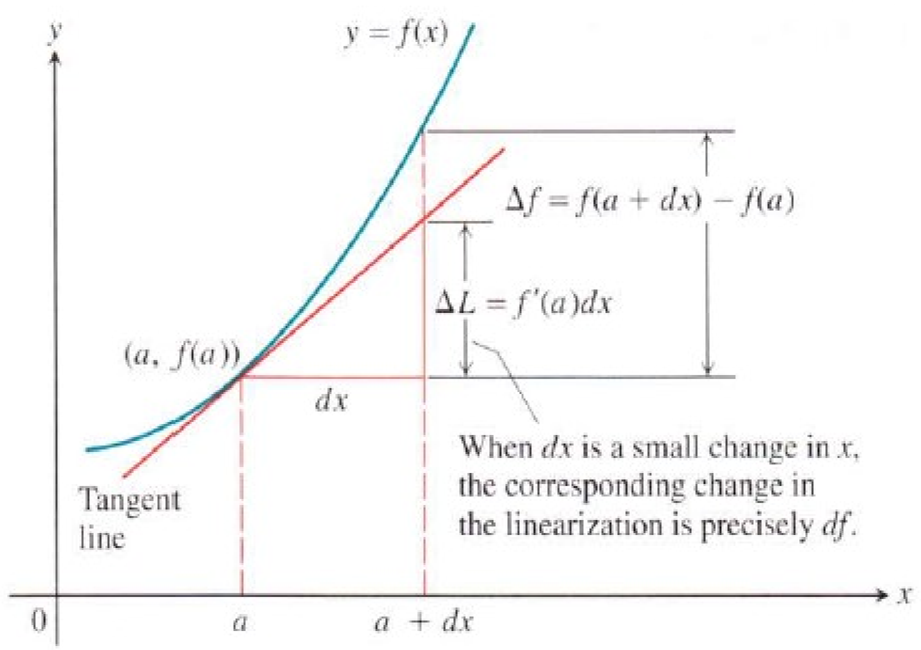
Calc 12 - Chp 4.5 Part 2

Differentials. - useful for approximating errors.

Definition: Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is

We can use this to estimate change or error.

$$\Delta f = f(a + dx) - f(a).$$



Differential Estimate of Change.
Let $f(x)$ be differentiable at $x = a$. The approx change in the value of f when x changes from
to is

Absolute, Relative, and Percentage Change

As we move from a to a nearby point $a + dx$, we can describe the change in f in three ways:

	True	Estimated
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a)\,dx$
Relative change	$\dfrac{\Delta f}{f(a)}$	$\dfrac{df}{f(a)}$
Percentage change	$\dfrac{\Delta f}{f(a)} \times 100$	$\dfrac{df}{f(a)} \times 100$

eg) #32) $S(r) = 4\pi r^2$

eg) #38) a) $A(r) = \pi r^2$

b)

eg) #40) $V(s) = s^3$

eg) #44) Working Backwards
$V(h) = \pi h^3$

HW: pp. 230-231: 31-41 odds, 46
Challenge: 43, 45

Calc 12 - Chp 4.6

Note Title 2013-09-23

Related Rates - putting everything together.

Simple example: A balloon is being inflated at $100\ cm^3/s$. How fast is the radius increasing when the diameter is 50 cm?

Write what's given:

Want dr/dt. What is the relationship between V & r?

Trig example: An actress walks across the front of the stage at 1 m/s. A spotlight follows her from 10 m away to the front of the stage. What rate is the light rotating when she is 5 m from the perpendicular?

Draw a diagram. What's given:

Another example: A water trough is 10 m long and a cross section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and a height of 50 cm. If the trough is being filled at a rate of $.2\ m^3/min$, how fast is the water level rising when the water is 30 cm deep?

Draw a diagram.

What's given:

Leave all answers as rationals or radicals.

HW: pp. 237-240: 1, 3, 7, 11, 13, 17, 21, 23, 31, 12
Challenge: 33, 35