Note Title 2013-09-23

Calc 12 - Chp 6.1

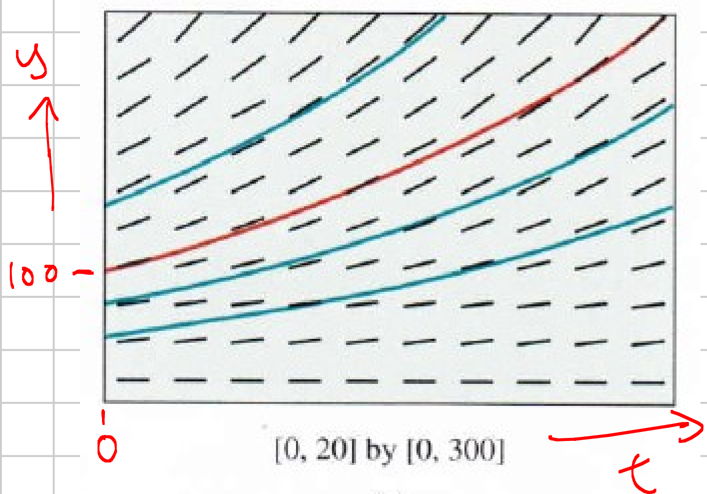
Antiderivatives and Slope Fields.

Definition **Slope Field or Direction Field** D.E.

A **slope field** or **direction field** for the first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

is a plot of short line segments with slopes $f(x, y)$ for a lattice of points (x, y) in the plane.



[0, 20] by [0, 300]

(b)

D E: $\frac{dy}{dt} = .056y$

I C: $y(0) = 100$

Similar to $y = -2x + b$. Need 'b' or a point as initial condition to get a specific solution.

$y(t) = C \cdot e^{.056t}$

$y(0) = 100 = Ce^{0}$

$C = 100$

$y(t) = 100 \cdot e^{.056t}$

Definition **Indefinite Integral**

The set of all antiderivatives of a function $f(x)$ is the **indefinite integral** of f with respect to x and is denoted by

$$\int f(x)\, dx.$$

We looked at antiderivatives in Chp 4.2.

$F'(x) = f(x)$ & if $f'(x) = g'(x)$ then $f(x) = g(x) + C$

So, ← constant of integration

1. (a) $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$ $\qquad \displaystyle\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,\ n \neq -1$

(b) $\displaystyle\int \frac{dx}{x} = \ln|x| + C$ $\qquad \displaystyle\frac{d}{dx}\ln|x| = \frac{1}{x}$

6) $\displaystyle\int \frac{dx}{ax+b} = \frac{\ln|ax+b|}{a} + C$

2. $\displaystyle\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$ $\qquad \displaystyle\frac{d}{dx}\left(\frac{e^{kx}}{k}\right) = e^{kx}$

3. $\displaystyle\int \sin kx\,dx = -\frac{\cos kx}{k} + C$ $\qquad \displaystyle\frac{d}{dx}\left(-\frac{\cos kx}{k}\right) = \sin kx$

4. $\displaystyle\int \cos kx\,dx = \frac{\sin kx}{k} + C$ $\qquad \displaystyle\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$

5. $\displaystyle\int \sec^2 x\,dx = \tan x + C$ $\qquad \displaystyle\frac{d}{dx}\tan x = \sec^2 x$

90) $\displaystyle\int \sec^2 ax\,dx = \frac{\tan ax}{a} + C$

6. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$ $\qquad \displaystyle\frac{d}{dx}(-\cot x) = \csc^2 x$

91) $\displaystyle\int \csc^2 ax\,dx = \frac{-\cot ax}{a} + C$

7. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$ $\qquad \displaystyle\frac{d}{dx}\sec x = \sec x \tan x$

94) $\displaystyle\int \sec^n ax \tan ax\,dx = \frac{\sec^n ax}{na} + C$

8. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$ $\qquad \displaystyle\frac{d}{dx}(-\csc x) = \csc x \cot x$

95) $\displaystyle\int \csc^n ax \cot ax\,dx = -\frac{\csc^n ax}{na} + C$

Note that 2, 3, 4 have an additional constant 'k'

eg) $\displaystyle\int x^{3/4}\,dx =$

eg) $\displaystyle\int \frac{1}{\sqrt[5]{x}}\,dx =$

eg) $\displaystyle\int e^{-5/2x}\,dx =$

eg) $\int \sin \frac{x}{4}\, dx =$

Try: $\int \frac{1}{\sqrt[3]{x}}\, dx =$

Try: $\int \cos 5x\, dx =$

Properties of Indefinite Integrals

Let k be a real number.

1. *Constant Multiple Rule:* $\int k f(x)\, dx = k \int f(x)\, dx$

   If $k = -1$, *then:* $\int -f(x)\, dx = -\int f(x)\, dx$

2. *Sum and Difference Rule:* $\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

eg) $\int e^{2x} - \sin \frac{x}{3} - x^2\, dx$

Solving Separable D.E. – these are equations that we can isolate one variable to LHS and the other variable to RHS. Then integrate both sides.

eg) Find the displacement function given:

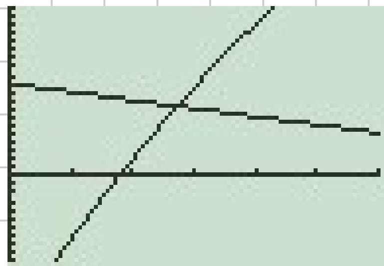
$v = \frac{ds}{dt}$, $a = \frac{dv}{dt} = -9.8 \text{ m/s}^2$

IC: $v(1) = 100$, $s(2) = 20$

Check solution on graphing calc.

```
Plot1 Plot2 Plot3
\Y1=-4.9X²+109.8
X-180
\Y2=nDeriv(Y1,X,
X)
\Y3=
\Y4=
\Y5=
```

```
WINDOW
 Xmin=0
 Xmax=6
 Xscl=1
 Ymin=-100
 Ymax=200
 Yscl=10
 Xres=1
```



X	Y1	Y2
0	-180	109.8
1	-75.1	100
2	20	90.2
3	105.3	80.4
4	180.8	70.6
5	246.5	60.8
6	302.4	51

X=0

HW: pp. 312-313: 7-17 odds, 31-37 odds, 34
Challenge: 25, 57

Calc 12 - Chp 6.2

Note Title 2013-09-23

Integration by Substitution. - all about seeing patterns!

Chain Rule: $\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}$

$$\int \left(u^n \frac{du}{dx}\right) dx = \frac{u^{n+1}}{n+1} + C$$

Simple eg) $\int (x-3)^2 \, dx$

Medium eg) $\int \sqrt{3x-5} \, dx$

Note never substitute 'u' without 'du'.
eg) $\int \sqrt{u} \, dx$ - integrating wrt x but no x's!

Don't forget to substitute original variable back in, otherwise marks will be lost!

Medium eg) $\int \frac{1}{\sin^2 3x}\,dx$

no antiderivative for $\sin^2 u$

Hard eg) $\int \sin x \cos^6 x\, dx$

Substitution in Definite Integrals

Substitute $u = g(x)$, $du = g'(x)\,dx$, and integrate with respect to u from $u = g(a)$ to $u = g(b)$.

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

must change limits of integration because of subst!

Med eg) Solve $\int_0^4 \sqrt{2x+1}\,dx$

Hard eg) $\int_0^{\pi/4} \tan x \sec^2 x \, dx$

Separable D.E.

$$\frac{dy}{dx} = g(x)h(y)$$

$$\frac{dy}{h(y)} = g(x)\,dx$$

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

eg) Solve $\frac{dy}{dx} = 2x(1+y^2)e^{x^2}$

Hw: pp. 321-322 : 1-21 odds, 31, 39, 43, 44
Challenge: 41

Calc 12 - Chp 6.3

Integration by Parts - Reverse of product rule (almost).

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$ Product Rule

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

$$\int\left(u\frac{dv}{dx}\right)dx = \int\left(\frac{d}{dx}(uv)\right)dx - \int\left(v\frac{du}{dx}\right)dx$$

$$\int u\,dv = uv - \int v\,du$$

Simple eg) $\int x\sin x\,dx$

Simple eg) $\int_0^4 -xe^{-x}\,dx$

IBP twice or more

Medium eg) $\int x^2 e^x \, dx$

Hard eg) $\int e^x \sin x \, dx$

In general, monomials should be 'u' and exponentials and trigonometrics should be 'dv'. Use last example as a template when exponential and trigonometric.

HW: pp. 328-329: 1-11 odds, 15, 17

Please solve $\int z \ln z \, dz$

Challenge: 29

Exponential Growth and Decay

Law of Exponential Change - many real-life situations experience change that is proportional the amount that is present. eg.

- compound interest
- radioactive decay
- Newton's Law of Cooling
- resistance in a fluid
- bacterial growth
- depreciation
- discharging a capacitor
- damped oscillations
- chemical reaction rates
- amount of drug in the bloodstream.

These are all represented by the first order DE:

D.E: $\frac{dy}{dt} = ky$ I.C: $y = y_0$ when $t = 0$
$y = y_i$ when $t = t_i$

General Solution: $\frac{dy}{y} = k\,dt$

Specific Solution:

Radioactive Decay - instead of using a base of ½ for the exponential function, we normally keep 'e' as the base. We want a positive 'k' value for the function, so we write '-k'

If 'τ' is the half-life:

Don't use ½ as the base; this is for Pre-Calc. If you are unsure about your solution, you can use Pre-Calc to check your answer.

eg) Carbon dating: Carbon-14 has a half-life of 5700 years. Find the age of a sample that has 15% of its original C-14 proportion.

Newton's Law of Cooling

temp diff

$$\frac{dT}{dt} = -k(T - T_s)$$

surrounding temp, such as room temp.

time

$$y(t) = T(t) - T_s$$

eg) A cup of coffee has just been poured is at 97°C. After 10 mins, it cools to 90°C. Room temperature is 20°C. How long will it take for the coffee to cool to 70°C?

Not deriving continuously compounding interest rate. It is $A(t) = A_0 e^{rt}$. It is theoretical and never used.

HW: pp. 338-339: 11, 13, 15, 19, 21, 25, 27
Challenge: 23

Calc 12 - Chp 6.5

Note Title 2013-09-23

Population Growth

Exponential growth such as the bacteria model is sufficient for small population and small intervals of time. If a simulation runs long enough, numbers will "blow up." For instance if we use exponential growth for humans, it won't take long before the numbers exceed 30 billion. We know that there aren't enough resources for population to keep growing. A more accurate model is the logistic growth model which takes account of limited resources. Logistic Function:

DE: $\frac{dP}{dt} = k\frac{P}{M}(M-P)$

M is the carrying capacity

$k > 0$

Notice if $P > M$, this will reduce the population.

Solve:

eg) Biologists stock a lake with 400 fish and estimate the carrying capacity to be 10,000. The number of fish triple after the first year. Use the logistic equation to determine when there will be 8000 fish. Compare with exp growth.

Exponential Growth:

$$A(t) = A_0 e^{kt}$$
$$= 400 e^{kt}$$
$$A(1) = 1200 = 400 e^{k}$$
$$3 = e^{k}$$
$$k = \ln 3$$

$$A(t) = 400 e^{t \ln 3}$$
$$8000 = 400 e^{t \ln 3}$$
$$20 = e^{t \ln 3}$$
$$\ln 20 = t \ln 3$$
$$t = \frac{\ln 20}{\ln 3}$$
$$\approx 2.7 \text{ years.}$$

Graph it! Be careful with brackets on calculator and whether multiply signs are necessary.

Determining M and K. Use $\frac{dP/dt}{P} = k\left(1 - \frac{P}{M}\right)$.

Use algebra to rearrange.

#8) In Exercises 7 and 8, find k and the carrying capacity for the population represented by the logistic differential equation.

$$\frac{50}{P}\frac{dP}{dt} = 2 - \frac{P}{250}$$

HW: pp. 347-348: 3, 5, 7, 13, 17, 9-12
Challenge: 19