Calc 12 - Chp 8.1

Note Title 2013-09-23

L'Hopital's Rule - please learn on your own.

Tips: Read Theorem 1 & take notes

" Examples 1 & 2

" 8.1 Intro

" Theorem 1

" Theorem 2

" Example 3

" Theorem 2.

" Examples 4-6

Work thru Example 7, make sure you understand each step.

HW: pp. 423-424: 1-7 odds, 15-27 odds

If you have trouble, read example 7 again.

Work thru Examples 8-10.

HW: p. 424: 33-39 odds.

Theorem 1: L'Hopital's Rule (First Form)
Suppose that $f(a) = g(a) = 0$, that $f'(a)$ & $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

eg) $\lim_{x \to 1} \frac{\ln x}{x-1}$ $\qquad$ $\ln 1 = 0$ & $x - 1 = 0$

However there are more complicated limits, so we need something stronger:

Theorem 2: L'Hopital's Rule (Stronger Form)
Suppose that $f(a) = g(a) = 0$ or $f(a) = g(a) = \pm\infty$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Note this is different because it is recursive, so we can apply it as many times as necessary.

eg) $\lim_{x \to 0} \frac{\tan x - x}{x^3}$ $\qquad$ $\tan 0 - 0 = 0$ & $0^3 = 0$

eg) $\lim_{x \to \infty} \dfrac{x^2}{e^x}$ $\qquad$ $\infty^2 = \infty$ & $e^{\infty} = \infty$

Extra Forms: $\overset{f}{0} \cdot \overset{g}{\infty}$ & $\overset{j}{\infty} \quad \infty$

Convert to a usable form:

$$\frac{g}{1/f} \text{ or } \frac{f}{1/g}$$

$j-k$ factor or find common denominator so we get $0/0$ or ∞/∞

$j = m/n$ & $k = p/q$ so $j - k = \dfrac{m}{n} - \dfrac{p}{q} = \dfrac{mq - np}{nq}$

Note one of n or q might equal 1.

eg) $\lim_{x\to 0^+} x\ln x$ $\qquad 0^+ = 0 \qquad \ln 0^+ = -\infty$

eg) $\lim_{x\to (\pi/2)} (\sec x - \tan x)$ $\qquad \sec \pi/2 = \infty \qquad \tan \pi/2 = \infty$

More Extra Forms: $0^0, 1^\infty, \infty^0$. Use logarithms.

$y = f(x)^{g(x)} \qquad \ln y = g(x)\ln f(x)$

soln: $f(x)^{g(x)} = e^{g(x)\ln f(x)}$

eg) $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$

Calc 12 - Chp 8.2

Note Title 2013-09-23

Relative Rates of Growth

Definitions: for $f(x)$ & $g(x)$ positive & x sufficiently large

1. f grows faster than g (implies g grows slower than f) as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \quad \text{or} \quad \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$$

2. f and g grow at the same rate as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad L \neq 0 \text{ and finite}$$

You should notice that L'Hopital's Rule might come handy for this section.

eg) Show that e^x grows faster than x^2

eg) Show that x grows faster than $\ln x$.

Transitivity of Growth Rates.

If f grows at the same rate as g as $x \to \infty$ and g grows at the same rate as h as $x \to \infty$, then f grows at the same rate as h.

eg) Show $\sqrt{2x^3-5}$ and $(3\sqrt{x}+1)^3$ have the same growth rate.

Definition: f of Smaller Order than g, if

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then we write $f = o(g)$

and say "f is little-oh of g"

Definition: f of at Most the Order of g, if

$\lim_{x\to\infty} \frac{f(x)}{g(x)} \leq M$, then we write $f = O(g)$

and say "f is big-oh of g"

eg) Show $4\ln x = o(x^2)$

eg) Show $2x + \cos x = O(x)$

HW: p. 431: 1-23 odds (please show 20)

Calc 12 - Chp 8.3 Part 1

Note Title 2013-09-23

Improper Integrals

$\int_0^{\infty} e^{-x/3}\,dx =$

As you know, we can't just substitute ∞.

Definitions: As long as $f(x)$ is continuous on the interval

1. $\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$
2. $\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx$
3. $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$, where $c \in \mathbb{R}$

Note: These definitions do not imply that the improper integral has a finite value. If the limit equals $\pm\infty$, we say the integral diverges. If the limit has a finite value, we say it converges.

Continuing previous eg

$\int_0^{\infty} e^{-x/3}\,dx =$

eg) Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Use def 3, let $c=0$.

eg) Evaluate $\int_1^\infty \frac{2}{x}\,dx$

Theorem: $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p>1$ else divergent.

Another improper integral happens when the interval ends at a vertical asymptote.

Definitions:

If f is continuous on $[a,b)$, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

If f is continuous on $(a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

If f is continuous on $[a,c) \cup (c,b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Again, we must determine if the improper integral converges or diverges.

eg) Find $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ V.A. @ $x=2$

eg) Find $\int_0^3 \frac{1}{x-1}\,dx$ V.A. @ $x=1$

HW: p. 442: 1-21 odds.
(Please solve 14 in person)

Calc 12 - Chp 8.3 Part 2

Note Title 2013-09-23

Improper Integrals - Convergence Tests.

Some integrals are difficult to find, so to test for convergence, we look for an known integral to compare.

Theorem 3 - Direct Comparison Test.

Let f and g be continuous on $[a,\infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$, then

1. $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges.
2. $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.

This is like a one-sided Sandwich Theorem.

eg) Is $\int_0^\infty e^{-x^2}\,dx$ convergent?

Substitution won't work: $u=-x^2$

$du = \underbrace{-2x}_{\text{not there.}}dx$

$\int e^{-x}\,dx$ is easily integrable.

eg) Is $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ convergent?

We notice that $\frac{1+e^{-x}}{x} > \frac{1}{x}$,

Theorem 4: Limit Comparison Test

If f and g are continuous on $[a,\infty)$, $f, g > 0$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, $L \in \mathbb{R}$, then

$\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or both diverge

eg) Does $\int_1^\infty \frac{1}{x+x^2}\,dx$ converge?

Try comparing with $\int_1^\infty \frac{1}{x^2}\,dx$

HW: p. 442: 27-43 odds
(Please solve 38 in person)

Calc 12 - Chp 8.4

Note Title 2013-09-23

Partial Fractions and Integral Tables.

Most polynomial denominators are difficult to integrate directly, so partial fractions is another tool for our toolbox. We must solve a system of linear equations when using partial fractions.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

p. 445

1. Let $x - r$ be a linear factor of $g(x)$. Suppose $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:
$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$
Do this for each distinct linear factor of $g(x)$.
2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose $(x^2+px+q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:
$$\frac{B_1x + C_1}{x^2+px+q} + \frac{B_2x + C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx + C_n}{(x^2+px+q)^n}.$$
Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.
3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Simple eg) $\int \frac{1}{x^2-a^2}\,dx$ *

Med eg) $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$ *

Hard eg) $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

$$
\begin{array}{r|l}
 & x+1 \\
\hline
x^3-x^2-x+1 & x^4+0x^3-2x^2+4x+1 \\
 & x^4-x^3-x^2+x \\
\hline
 & 0 \quad x^3-x^2+3x+1 \\
 & \quad\; x^3-x^2-x+1 \\
\hline
 & \qquad\qquad 4x
\end{array}
$$

HW: p. 452: 1–13 odds, 17, 21, 25
(Please show 18 in person)